

ON THE NEWMAN SUM OVER MULTIPLES OF A PRIME WITH A PRIMITIVE OR SEMIPRIMITIVE ROOT 2

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ABSTRACT. We prove that if $S_p(x)$ is the Newman sum over p -multiples where p is a prime with a primitive (semiprimitive) root 2 then $S_p(2^p x) = pS_p(2x)$ ($S_p(2^p x) = (-1)^{\frac{p-1}{2}} pS_p(2x)$). We consider the case of $p = 17$ as well.

1. INTRODUCTION

Consider for $x, m, l \in \mathbb{N}$, $l \in [0, m-1]$, the Newman sum

$$S_m(x) = S_{m,0}(x) = \sum_{0 \leq n < x, n \equiv 0 \pmod{m}} (-1)^{\sigma(n)},$$

where $\sigma(n)$ is the number of 1's in the binary expansion of n .

In [6] we proved that if 2 is a primitive root of a prime p then $S_p(2^p) = p$. Now we prove a more general relation.

Theorem 1. *If 2 is a primitive root of a prime p then for any natural x*

$$(1) \quad S_p(2^p x) = pS_p(2x).$$

Furthermore, let 2 be not a primitive root of an odd prime p . We say that 2 is a *semiprimitive root* of p if 2 has the order $\frac{p-1}{2}$ modulo p and the congruence $2^x \equiv -1 \pmod{p}$ is not solvable.

Example 1. *2 has the order $\frac{p-1}{2}$ for $p = 7, 17, 23$ but only for $p = 17$ the congruence $2^x \equiv -1 \pmod{17}$ has a solution ($x = 4$). Therefore, by the definition, 2 is a semiprimitive root for 7, 23 (but not for 17).*

Note that, if 2 is a semiprimitive root of p then for every integer $a \in [1, p-1]$ there exist $j = j_a$ such that either $a \equiv 2^j \pmod{p}$ or $a \equiv -2^j \pmod{p}$.

Our statement similar to Theorem 1 in the case of 2 is a semiprimitive root of p is the following.

Theorem 2. *If 2 is a semiprimitive root of a prime p then for any natural x*

$$(2) \quad S_p(2^p x) = (-1)^{\frac{p-1}{2}} p S_p(2x).$$

Theorems 1 and 2 show that if 2 is primitive or semiprimitive root of an odd prime p then

$$(3) \quad S_p(x) = O\left(x^{\frac{\ln p}{(p-1)\ln 2}}\right).$$

and open a way similar to [4] to get the sharp estimates for $S_p(x)$ in considered cases, i.e. to generalize the Coquet's theorem (see[1], p.98-99).

On the other hand, (3) makes more precise the remainder term of the Gelfond theorem in the considered case:

$$G_p^{(i)}(x) = \sum_{0 \leq n < x, n \equiv 0 \pmod{p}, \sigma(n) \equiv i \pmod{2}} 1 = \frac{x}{2p} + O\left(x^{\frac{\ln p}{(p-1)\ln 2}}\right), i = 0, 1,$$

instead of $O\left(x^{\frac{\ln 3}{\ln 4}}\right)$ in [3]. Moreover, these estimates are unimprovable.

2. PROOF OF THEOREM 1

We again use the formula (cf.[2])

$$(4) \quad S_p(2^k) = \frac{1}{p} \sum_{l=1}^{p-1} \prod_{j=0}^{k-1} (1 - \omega_p^{l2^j}),$$

where $\omega_p \neq 1$ is a primitive root of 1 of the power p . By (4) we have also for $k \geq p$

$$(5) \quad S_p(2^{k-p+1}) = \frac{1}{p} \sum_{l=1}^{p-1} \prod_{j=0}^{k-p} (1 - \omega_p^{l2^j}).$$

Let us consider the quotient

$$Q = \frac{\prod_{j=0}^{k-1} (1 - \omega_p^{l2^j})}{\prod_{j=0}^{k-p} (1 - \omega_p^{l2^j})} = \prod_{k-(p-1)}^{k-1} (1 - \omega_p^{l2^j}), \quad l = 1, 2, \dots, p-1.$$

Considering here the substitutions $j - k + p = t$, $l_1 = l \cdot 2^{k-p}$ we find

$$(6) \quad Q = \prod_{t=1}^{p-1} \left(1 - \omega_p^{l, 2^t}\right)$$

Since 2 is a primitive root of p then independently on l

$$(7) \quad Q = \prod_{t=1}^{p-1} (1 - \omega_p^t)$$

Note that

$$\prod_{t=1}^{p-1} (x - \omega_p^t) = \frac{x^p - 1}{x - 1} = 1 + x + \dots + x^{p-1}.$$

Therefore, by (7) $Q = p$ and according to (4)-(5)

$$S_p(2^p \cdot 2^{k-p}) = p S_p(2 \cdot 2^{k-p}).$$

Thus, for $x = 2^n$, $n \geq 0$, we have

$$(8) \quad S_p(2^p x) = p S_p(2x).$$

Finally, using the additive properties of S_p as in [4] we obtain (8) for any nonnegative integer x . ■

As well, note that if $S_p([x, y])$ denotes the difference $S_p(y) - S_p(x)$ then we have

$$(9) \quad S_p([2^p x, 2^p y]) = p S_p([2x, 2y]).$$

3. PROOF OF THEOREM 2.

Since 2 is a semiprimitive root of p then instead of (6) independently of l_1 we have

$$\begin{aligned} Q &= \prod_{j=1}^{\frac{p-1}{2}} \left(1 - \omega_p^{2^j}\right)^2 = (-1)^{\frac{p-1}{2}} \prod_{j=1}^{\frac{p-1}{2}} \left(1 - \omega_p^{2^j}\right) \prod_{j=1}^{\frac{p-1}{2}} \left(\omega_p^{2^j} - 1\right) = \\ &= (-1)^{\frac{p-1}{2}} \omega^{2+2^2+\dots+2^{\frac{p-1}{2}}} \prod_{j=1}^{\frac{p-1}{2}} \left(\left(1 - \omega_p^{2^j}\right) \left(1 - \omega_p^{-2^j}\right)\right) = \\ &= (-1)^{\frac{p-1}{2}} \prod_{k=1}^{p-1} (1 - \omega_p^k) = (-1)^{\frac{p-1}{2}} p. \end{aligned}$$

Now in this case similar to (8) we obtain (2) and the following relation

$$S_p([2^p x, 2^p y]) = (-1)^{\frac{p-1}{2}} p S_p([2x, 2y]). \blacksquare$$

4. CASE OF $p = 17$

Now we give a relation for the first number of the Drmota-Skalba primes [2] $p = 17$ for which 2 is neither primitive nor semiprimitive root.

Theorem 3.

$$S_{17}(2^{17}x) = 34S_{17}(2^9x) - 17S_{17}(2x), \quad x \in \mathbb{N}.$$

In particular, in the case of $x = 1$ we have

$$S_{17}(2^{17}) = 34S_{17}(2^9) - 17S_{17}(2) = 31 \cdot 21 - 17 = 697.$$

Using Theorem 3 as in [4] it could be proved that

$$S_{17}(x) = O(x^\alpha)$$

with $\alpha = \frac{\ln(17+4\sqrt{17})}{\ln 256} = 0.633220353\dots$. It is essentially more than $\frac{\ln 17}{16 \ln 2}$ but less than $\frac{\ln 3}{2 \ln 2}$. *Is it true for the further Fermat primes the relation*

$$S_p(2^p x) = 2p S_p(2^{\frac{p+1}{2}} x) - p S_p(2x)?$$

Unfortunately, our method ([4]) for receiving such relations is too tiring and does not give anything for large modulus. It is very interesting to understand if the cases when 2 is (semi)primitive root of p are all the cases when we have a binomial relation of the form $S_p(2^p x) = ap S_p(2x)$?

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